Fractional Kinetics

→ More systematic, though sometimes more obscure, than ~~E~~TRW

→ FK relevant if singular domains which are "sticky" on phase space

and

→ Phase space 'rough', or fractal, so must treat roughness / fractal structure. → self-similar

→ Key is critical exponents — scaling!
idea ∂_t^{α}, ∂_x^{β}

critical exponents generalize
$\partial_t \langle x^2 \rangle \sim D$
$\langle x^2 \rangle \sim Dt$
$\langle x^2 \rangle \sim t^{\mu}$

To Derive:

— phenomenological derivation of eqn.

— relation between exponents.

Now $P(x,t) = W(x, x_0, t-t_0) = W(x, x_0, t)$

is transition probability

$\Delta_t P \rightarrow$ shift in P along t, by Δ_t

normal kinetics

$$\Delta_t P(x,t) = P(x, t+\Delta t) - P(x,t)$$

$$= \underbrace{\Delta t \frac{\partial P}{\partial t}}_{\text{dt FP}} + \ldots$$

Now, if fractal time:

$$\Delta_t^\beta P = \frac{\partial^\beta P}{\partial t^\beta} \underbrace{(\Delta t)^\beta}_{\text{fractional alternative}} + \ldots (\Delta t)^{\beta_1}, \quad \boxed{0 < \beta \leq 1}$$

$$\beta_1 > \beta$$

where:

$$(-i\omega)^\beta P(\omega) = \mathcal{L}\left\{ \frac{\partial^\beta P}{\partial t^\beta} \right\} \quad \boxed{\beta < 1}$$

see Z for techn. details

RHS singular as $\Delta t \to 0$!

N.B. Fractional time can obviously represent sticking, flights etc.

Similarly, proceed to space

$$\Delta_x^\alpha P = \int dy\, W(x,y,\Delta t)\, P(y,t) - P(x,t) + o(\Delta t)^{\beta_1}$$

so, can represent conservation of P particles due steps in space, time as:

$$\boxed{\Delta_t^{\beta} P(x,t) = \Delta_x^{\alpha} P(x,t) + O(\Delta t^{\beta_3})}$$

$\min \beta_2, \beta_3$

or

$$\boxed{\lim \frac{1}{(\Delta t)^{\beta}} \Delta_t^{\beta} P(x,t) = \lim_{\Delta t \to 0} \frac{\Delta_x^{\alpha} P(x,t)}{(\Delta t)^{\beta}}}$$

- Formal expr FK — continuity eqn pdf.
- how calc RHS. ?

Now, using expansion for Δ_t^{β}:

$$\frac{\partial^{\beta} P}{\partial t^{\beta}} = \lim_{\Delta t \to 0} \frac{1}{(\Delta t)^{\beta}} \left\{ \int dy\, w(x,y,\Delta t) P(y,t) - P(x,t) \right\}$$

$0 < \beta \leq 1$

still no swindles re: W

Now assume expansion of W:
($\Delta t \to 0$)

cancels

$\sim (\Delta x)$

$$W(x, y, \Delta t) = \delta(x-y) + A(y, \Delta t)\, \delta^{\alpha}(x-y)$$

$$+ B(y, \Delta t)\, \delta^{\alpha_1}(x-y)$$

$\sim \angle (\Delta x)^2)$

$$0 < \alpha < \alpha_1 \leq 2$$

$\alpha = 1$
$ 2$
for standard

- finite # terms in exp. (cf. F-P)

- $A(y, \Delta t)$, $B(y, \Delta t)$ indep. $P(x,t)$
  i.e. linear eqn., specify transition prob

Point: $W \to$ local features dynamics

$P \to$ strongly non-local features
$t \to \infty$

so

A, B indep P $\Rightarrow$ independence of local transitions from long time behavior..

Can then show:

$$\langle |\Delta x|^{\alpha_1} \rangle = \int dx\, |x-y|^{\alpha_1} W(x,y,\Delta t)$$

$$= \int dx\, |x-y|^{\alpha_1} \Big[\cancel{\delta(x-y)} + A\, \cancel{\partial^{\alpha}(x-y)} + B\, \partial^{\alpha_1}(x-y) \Big]$$

leaves $(x-y)^{\alpha_1-\alpha}$ ibp $\to 1$

$\Rightarrow$

$$\boxed{\langle |\Delta x|^{\alpha_1} \rangle = \Gamma(1+\alpha_1)\, B(y,\Delta t)}$$

conclude at vs time rel'n

— A not so simple...

$$\int dy\, W(x,y,\Delta t) =$$

$$\int dy\, \cancel{W} = \cancel{1} + \frac{\partial^{\alpha} A}{\partial (-x)^{\alpha}} + \frac{\partial^{\alpha_1} B}{\partial (-x)^{\alpha_1}}$$

by IBP

$\Rightarrow$

$$\boxed{\frac{\partial^{\alpha} A(x)}{\partial (-x)^{\alpha}} + \frac{\partial^{\alpha_1} B(x)}{\partial (-x)^{\alpha_1}} = 0}$$

$\xrightarrow{\text{skip}}$ Analogue:

$$\frac{\langle \Delta x \rangle}{\Delta t} = \frac{1}{2} \frac{\partial D}{\partial x}$$

where:

$$A(x) = \lim_{\Delta t \to 0} \frac{A(x, \Delta t)}{(\Delta t)^{\beta}}$$

$$B(x) = \lim_{\Delta t \to 0} \frac{B(y, \Delta t)}{(\Delta t)^{\beta}}$$

$$= \frac{1}{\Gamma(\beta + \alpha_1)} \lim_{\Delta t \to 0} \frac{\langle |\Delta x|^{\alpha_1} \rangle}{(\Delta t)^{\beta}}$$

Recall in standard kinetics,

$$A = \frac{1}{2} \frac{\partial}{\partial y} B$$

$$\frac{\langle \Delta x \rangle}{\Delta t} = \frac{1}{2} \frac{\partial}{\partial x} D$$

exercise $\rightarrow$ show

so, FKE is

$$\boxed{\frac{\partial^{\beta}}{\partial t^{\beta}} P(x,t) = \lim_{\Delta t \to 0} \frac{1}{(\Delta t)^{\beta}} \left\{ \int dy \left[W(x, y, t + \Delta t) - \delta(x-y) \right] P(x,t) \right\}}$$

so expanding as before, and using
A, B defs:

$$\boxed{\frac{\partial^\beta}{\partial t^\beta} P(x,t) = \frac{\partial^\alpha}{\partial(-x)^\alpha}\left(A(x)\, P(x,y)\right) + \frac{\partial^{\alpha_1}}{\partial(-x)^{\alpha_1}}\left(B(x)\, P(x,y)\right)}$$

F. F-P K — Fractional diffq.

for $\alpha_1 = \alpha + 1$

$$\frac{\partial^\beta p}{\partial t^\beta} = -\frac{\partial^\alpha}{\partial(-x)^\alpha}\left(B\frac{\partial p}{\partial x}\right)$$

$\rightarrow \alpha = 1,\quad B = D/2 \rightarrow$ diffeq

→ Can consider simplified cases:

i.e.

$$\frac{\partial^\beta P(x,t)}{\partial t^\beta} = \frac{\partial^\alpha}{\partial(-x)^\alpha}\left(A(x)\,P(x,t)\right) + \frac{\partial^{\alpha_1}}{\partial(-x)^{\alpha_1}}\left(B(x)\,P(x,t)\right)$$

$\alpha_1 = 1+\alpha$

$$\frac{\partial^\beta P}{\partial t^\beta} = -\frac{\partial^\alpha}{\partial(-x)^\alpha}\left(B\frac{\partial P}{\partial x}\right)$$

⇒

So if B negligible:

$$\boxed{\frac{\partial^\beta P}{\partial t^\beta} = \frac{\partial^\alpha}{\partial|x|^\alpha}(AP)}$$

$\alpha = 2$

$0<\beta<1$

⇒ $\beta = 1$

$\alpha = 2$

⇒ normal diffn

→ eqn of fractional BM

$|x|^\alpha \sim t^\beta$

Likewise, for:

$$\boxed{\frac{\partial P}{\partial t} = \frac{\partial^{\alpha}}{\partial |x|^{\alpha}} (A P)} \qquad \left[\begin{array}{l} \beta = 1 \\ 1 < \alpha < 2 \end{array} \right.$$

→ Levy process

$\boxed{\beta, \alpha, \alpha_1}$ → critical exponents → how extract?

↓

phase space structure

→ Conditions for FKE

Physical systems demand constraints:

- interval: need define space-time intervals to ensure integrability of P

- positivity: $P(x,t) \geq 0$

⇒ constrains critical exponent.

⇒ need: $\begin{cases} 0 < \alpha \leq 2 & \text{Levy} \\ 0 < \beta \leq 1, \ 0 < \alpha \leq 2 \end{cases}$

- Intermediate asymptotics — different asymptotics, diff^nt windows
- Definition fractional integro-dif.

Moments / Transport → contains physics obs.

$P(x,t)$ solves FKE

Moments:

$$\langle |x|^{\sigma} \rangle = \int dx\, |x|^{\sigma} P(x,t) \quad \begin{cases} \text{— macro observable} \\ \text{— defines transport} \end{cases}$$

if A = const in:

$$\frac{\partial^{\beta}}{\partial t^{\beta}} P = \frac{\partial^{\alpha}}{\partial |x|^{\alpha}} (A P)$$

then

$$\int |x|^{\alpha} \frac{\partial^{\beta} P}{\partial t^{\beta}} = \int |x|^{\alpha} \frac{\partial^{\alpha}}{\partial |x|^{\alpha}} (A P)$$

so

$$\boxed{\frac{\partial^\beta \langle |x^\alpha| \rangle}{\partial t^\beta} = A \int dx\, P\, \frac{\partial^\alpha}{\partial |x|^\alpha} |x|^\alpha = A\, \Gamma(1+\alpha)}$$

$$\Rightarrow \boxed{\langle |x^\alpha| \rangle = \frac{A\, \Gamma(1+\alpha)}{\Gamma(1+\beta)}\, t^\beta}$$

$$\sim (A\, \#)\, t^\beta$$

$\Rightarrow$ so, for self-sim soln:

$$\boxed{\langle |x| \rangle \sim t^{\beta/\alpha} \sim t^{\mu/2}}$$

which gives:

set by ratio of critical exps

$$\boxed{\mu = 2\beta/\alpha \rightarrow \underline{\text{transport exponent}}}$$

Meaning ?

Now, for $\langle x^2 \rangle < \infty$

$\Rightarrow \boxed{\langle x^2 \rangle \sim t^{\mu}}$

$\boxed{\begin{array}{c} \mu < 2 \\ \mu = 2\beta/\alpha \end{array}}$

$\mu > 1 \rightarrow$ superdiffn

$\mu < 1 \rightarrow$ subdiffusion

set by α, β exponents

Now, also recall:

$x^2 \sim Dt$

$x \sim \sqrt{Dt}$

$\dot{x} \sim \sqrt{D}/\sqrt{t}$ $\Rightarrow$ diffn exhibits false high speed at short t

but $\langle x^2 \rangle \sim Dt \sim \acute{D}\tau_{ac}t$

$\sim \langle (\dot{x})^2 \rangle \tau_{ac} t$

so D ill defined for $t \lesssim \tau_{ac}$.

have similar state here: (in FK)

Can take:

$$\frac{(dx)^\alpha}{(dt)^\beta} \sim \left(\frac{dx}{dt}\right)^\alpha dt^{\alpha-\beta}$$

$$\sim v^\alpha dt^{\alpha-\beta}$$

but:

<u>normal diffn</u>

$$\lim_{\Delta t\to 0} \frac{1}{\Delta t} \langle\langle \Delta x \rangle\rangle \equiv A(x)$$

and with α, β

$$\langle dx^\alpha \rangle \sim A\, (dt)^\beta$$

$\Rightarrow$

$$\frac{\langle dx^\alpha \rangle}{\langle (dt)^\beta \rangle} \sim \underset{\substack{\downarrow \\ \to 0}}{\overset{\substack{\to\infty! \\ \uparrow}}{v^\alpha}\, dt^{\alpha-\beta}} \sim \overset{\substack{\text{const.} \\ \downarrow}}{A}$$

$$\underset{>0}{\alpha-\beta} = \alpha(1-\mu/2) \qquad \underline{\mu < 2}$$

$$V^{\alpha}\, dt^{\alpha-\beta} \sim A$$

$\Rightarrow V \to \infty$ as $dt \to 0$

similar to short time, high speed requirement in diffn.

$\therefore$ have short time ($\sim T_{eu}$) cut-off

Now, $\left.\begin{array}{c}\beta = 1 \\ 1 < \alpha < 2\end{array}\right\}$ Levy conditions

$\mu = 2\beta/\alpha = 2/\alpha < 2$, as higher moments $\mu \geq 2$ diverge

$\boxed{1 < \mu < 2} \rightarrow$ superdiffusive (sub?)

So $\boxed{1 < \alpha < 2}$

$\boxed{0 < \alpha < 1}$ unclear why

$\rightarrow$ Origin of Critical Exponents,

Where α, β etc from?? $\rightarrow$ phase space struct

Recall standard map (OH 7):

$$p_{n+1} = p_n + \sin\psi_n$$

$$\psi_{n+1} = \psi_n + 2\pi M/u_{n+1}$$

$\Rightarrow$ kick in velocity

$$\boxed{\begin{aligned} I_{n+1} &= I_n + K\sin\theta_n \\ \theta_{n+1} &= \theta_n + I_n \end{aligned}}$$

$\hookrightarrow$ strength param. $\rightarrow$ causal

both mod 2π

stream

$\left(K\sin\theta_l = 2\pi l\right)$

$\downarrow$ accelerator mode

$\left(D_{QL} = K^2/2\right)$

peaks extend as $t\to\infty$

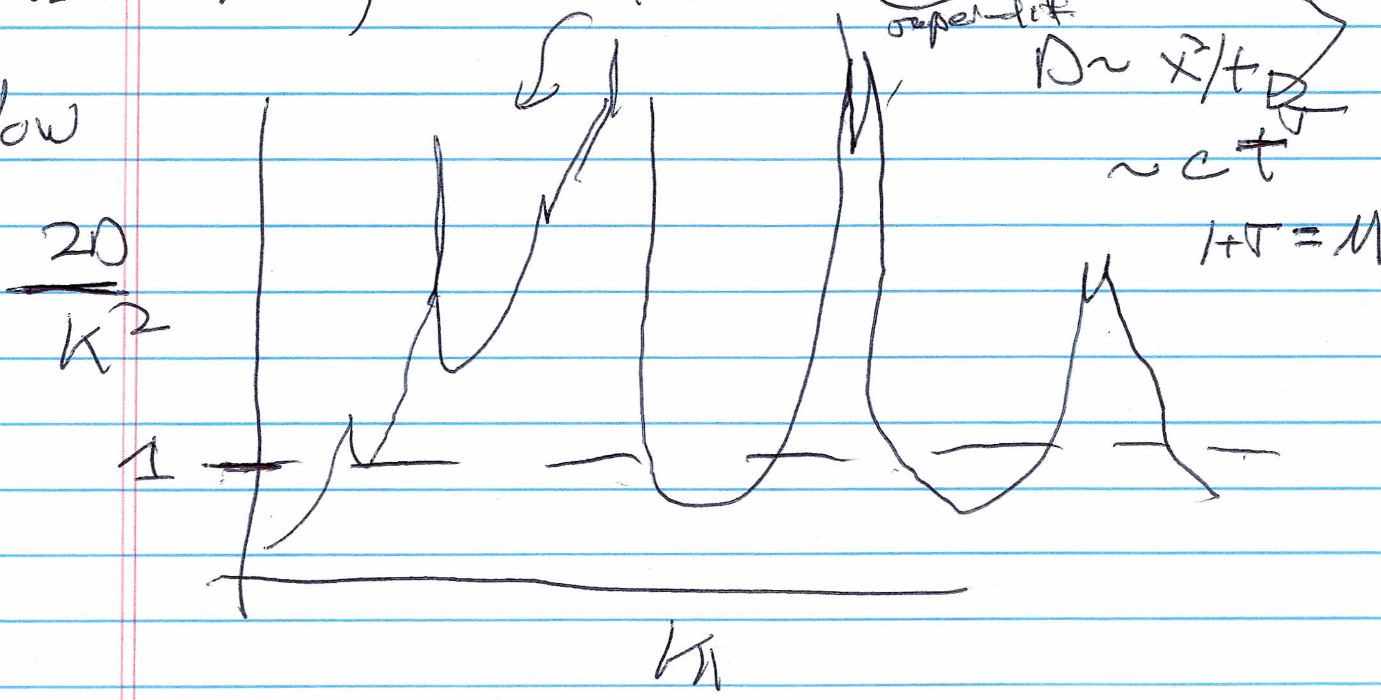
super-diff

$D\sim x^2/t$

$\sim ct$

$1+\sigma = M$

Now



- Now, sims show $D(K) \neq K^2/2 = D_{QL}$
- indeed → high peak in effective D

$$\langle p^2 \rangle \sim \text{const}\; t^{\mu_p}, \quad \mu_p > 1$$

super-diffusive

$\mu > 1 \Rightarrow \exists\, \alpha, \beta \quad \underline{\mu = 2\beta/\alpha}$

- many peaks, many sets exponents
- exponents localized in domains,

Punchline:

- in standard map, accelerator modes underpin super-diffusive scal,
- special values of K underpin flights
- no universal behavior. Each range flights has own crit. exp.